

An extremal problem on potentially $K_{p,1,1}$ -graphic sequences *

Chunhui Lai

Department of Mathematics

Zhangzhou Teachers College, Zhangzhou

Fujian 363000, P. R. of CHINA.

e-mail: zjlaichu@public.zzptt.fj.cn

Abstract

A sequence S is potentially $K_{p,1,1}$ graphical if it has a realization containing a $K_{p,1,1}$ as a subgraph, where $K_{p,1,1}$ is a complete 3-partite graph with partition sizes $p, 1, 1$. Let $\sigma(K_{p,1,1}, n)$ denote the smallest degree sum such that every n -term graphical sequence S with $\sigma(S) \geq \sigma(K_{p,1,1}, n)$ is potentially $K_{p,1,1}$ graphical. In this paper, we prove that $\sigma(K_{p,1,1}, n) \geq 2[(p+1)(n-1)+2]/2$ for $n \geq p+2$. We conjecture that equality holds for $n \geq 2p+4$. We prove that this conjecture is true for $p=3$.

Key words: graph; degree sequence; potentially $K_{p,1,1}$ -graphic sequence

AMS Subject Classifications: 05C07, 05C35

1 Introduction

If $S = (d_1, d_2, \dots, d_n)$ is a sequence of non-negative integers, then it is called graphical if there is a simple graph G of order n , whose degree sequence $(d(v_1), d(v_2), \dots, d(v_n))$ is precisely S . If G is such a graph then G is said to realize S or be a realization of S . A graphical sequence S is potentially H graphical if there is a realization of S containing H as a subgraph, while S is forcibly H graphical if every realization of S contains H as a subgraph. Let $\sigma(S) = d(v_1) + d(v_2) + \dots + d(v_n)$, and $[x]$ denote the largest integer less than or equal to x . We denote $G+H$ as the graph with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$. Let K_k , and C_k denote

*Project Supported by NNSF of China(10271105), NSF of Fujian, Science and Technology Project of Fujian, Fujian Provincial Training Foundation for "Bai-Quan-Wan Talents Engineering", Project of Fujian Education Department and Project of Zhangzhou Teachers College.

a complete graph on k vertices, and a cycle on k vertices, respectively. Let $K_{p,1,1}$ denote a complete 3-partite graph with partition sizes $p, 1, 1$.

Given a graph H , what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted $ex(n, H)$, and is known as the Turán number. This problem was proposed for $H = C_4$ by Erdős [3] in 1938 and in general by Turán [12]. In terms of graphic sequences, the number $2ex(n, H) + 2$ is the minimum even integer l such that every n -term graphical sequence S with $\sigma(S) \geq l$ is forcibly H graphical. Here we consider the following variant: determine the minimum even integer l such that every n -term graphical sequence S with $\sigma(S) \geq l$ is potentially H graphical. We denote this minimum l by $\sigma(H, n)$. Erdős, Jacobson and Lehel [4] showed that $\sigma(K_k, n) \geq (k-2)(2n-k+1) + 2$ and conjectured that equality holds. They proved that if S does not contain zero terms, this conjecture is true for $k = 3$, $n \geq 6$. The conjecture is confirmed in [5],[7],[8],[9] and [10].

Gould, Jacobson and Lehel [5] also proved that $\sigma(pK_2, n) = (p-1)(2n-2) + 2$ for $p \geq 2$; $\sigma(C_4, n) = 2\lceil \frac{3n-1}{2} \rceil$ for $n \geq 4$. Luo [11] characterized the potentially C_k graphic sequence for $k = 3, 4, 5$. Yin and Li [13] gave sufficient conditions for a graphic sequence being potentially $K_{r,s}$ -graphic, and determined $\sigma(K_{r,r}, n)$ for $r = 3, 4$. Lai [6] proved that $\sigma(K_4 - e, n) = 2\lceil \frac{3n-1}{2} \rceil$ for $n \geq 7$. In this paper, we prove that $\sigma(K_{p,1,1}, n) \geq 2[(p+1)(n-1) + 2]/2$ for $n \geq p+2$. We conjecture that equality holds for $n \geq 2p+4$. We prove that this conjecture is true for $p = 3$.

2 Main results.

Theorem 1. $\sigma(K_{p,1,1}, n) \geq 2[(p+1)(n-1) + 2]/2$, for $n \geq p+2$.

Proof. If $p = 1$, by Erdős, Jacobson and Lehel [4], $\sigma(K_{1,1,1}, n) \geq 2n$, Theorem 1 is true. If $p = 2$, by Gould, Jacobson and Lehel [5], $\sigma(K_{2,1,1}, n) = \sigma(K_4 - e, n) \geq \sigma(C_4, n) = 2\lceil (3n-1)/2 \rceil$, Theorem 1 is true. Then we can suppose that $p \geq 3$.

We first consider odd p . If n is odd, let $n = 2m+1$, by Theorem 9.7 of [2], K_{2m} is the union of one 1-factor M and $m-1$ spanning cycles $C_1^1, C_2^1, \dots, C_{m-1}^1$. Let

$$H = C_1^1 \cup C_2^1 \cup \dots \cup C_{\frac{p-1}{2}}^1 + K_1$$

Then H is a realization of $((n-1)^1, p^{n-1})$, where the symbol x^y stands for y consecutive terms x . Since $K_{p,1,1}$ contains two vertices of degree $p+1$ while $((n-1)^1, p^{n-1})$ only contains one integer $n-1$ greater than degree p , $((n-1)^1, p^{n-1})$ is not potentially $K_{p,1,1}$ graphic. Thus

$$\sigma(K_{p,1,1}, n) \geq (n-1) + p(n-1) + 2 = 2[(p+1)(n-1) + 2]/2.$$

Next, if n is even, let $n = 2m+2$, by Theorem 9.6 of [2], K_{2m+1} is the union of m spanning cycles $C_1^1, C_2^1, \dots, C_m^1$. Let

$$H = C_1^1 \cup C_2^1 \cup \dots \cup C_{\frac{p-1}{2}}^1 + K_1$$

Then H is a realization of $((n-1)^1, p^{n-1})$, and we are done as before. This completes the discussion for odd p .

Now we consider even p . If n is odd, let $n = 2m + 1$, by Theorem 9.7 of [2], K_{2m} is the union of one 1-factor M and $m - 1$ spanning cycles $C_1^1, C_2^1, \dots, C_{m-1}^1$. Let

$$H = M \cup C_1^1 \cup C_2^1 \cup \dots \cup C_{\frac{p-2}{2}}^1 + K_1$$

Then H is a realization of $((n-1)^1, p^{n-1})$, and we are done as before. Next, if n is even, let $n = 2m + 2$, by Theorem 9.6 of [2], K_{2m+1} is the union of m spanning cycles $C_1^1, C_2^1, \dots, C_m^1$. Let

$$C_1^1 = x_1 x_2 \dots x_{2m+1} x_1$$

$$H = (C_1^1 \cup C_2^1 \cup \dots \cup C_{\frac{p}{2}}^1 + K_1) - \{x_1 x_2, x_3 x_4, \dots, x_{2m-1} x_{2m}, x_{2m+1} x_1\}$$

Then H is a realization of $((n-1)^1, p^{n-2}, (p-1)^1)$. It is easy to see that $((n-1)^1, p^{n-2}, (p-1)^1)$ is not potentially $K_{p,1,1}$ graphic. Thus

$$\sigma(K_{p,1,1}, n) \geq (n-1) + p(n-2) + p-1 + 2 = 2[(p+1)(n-1) + 2]/2.$$

This completes the discussion for even p , and so finishes the proof of Theorem 1.

Theorem 2. For $n = 5$ and $n \geq 7$,

$$\sigma(K_{3,1,1}, n) = 4n - 2.$$

For $n = 6$, if S is a 6-term graphical sequence with $\sigma(S) \geq 22$, then either there is a realization of S containing $K_{3,1,1}$ or $S = (4^6)$. (Thus $\sigma(K_{3,1,1}, 6) = 26$.)

Proof. By theorem 1, for $n \geq 5$, $\sigma(K_{3,1,1}, n) \geq 2[(3+1)(n-1) + 2]/2 = 4n - 2$. We need to show that if S is an n -term graphical sequence with $\sigma(S) \geq 4n - 2$, then there is a realization of S containing a $K_{3,1,1}$ (unless $S = (4^6)$). Let $d_1 \geq d_2 \geq \dots \geq d_n$, and let G is a realization of S .

Case: $n = 5$, if a graph has size $q \geq 9$, then clearly it contains a $K_{3,1,1}$, so that $\sigma(K_{3,1,1}, 5) \leq 4n - 2$.

Case: $n = 6$, If $\sigma(S) = 22$, we first consider $d_6 \leq 2$. Let S' be the degree sequence of $G - v_6$, so $\sigma(S') \geq 22 - 2 \times 2 = 18$. Then S' has a realization containing a $K_{3,1,1}$. Therefore S has a realization containing a $K_{3,1,1}$. Now we consider $d_6 \geq 3$. It is easy to see that S is one of $(5^2, 3^4)$, $(5^1, 4^2, 3^3)$ or $(4^4, 3^2)$. Obviously, all of them are potentially $K_{3,1,1}$ -graphic. Next, if $\sigma(S) = 24$, we first consider $d_6 \leq 3$. Let S' be the degree sequence of $G - v_6$, so $\sigma(S') \geq 24 - 3 \times 2 = 18$. Then S' has a realization containing a $K_{3,1,1}$. Therefore S has a realization containing a $K_{3,1,1}$. Now we consider $d_6 \geq 4$. It is easy to see that $S = (4^6)$. Obviously, (4^6) is graphical and (4^6) is not potentially $K_{3,1,1}$ graphic. Finally, suppose that $\sigma(S) \geq 26$. We first consider $d_6 \leq 4$. Let S' be the degree sequence of $G - v_6$, so $\sigma(S') \geq 26 - 2 \times 4 = 18$. Then S' has a realization containing a $K_{3,1,1}$. Therefore S has a realization containing a $K_{3,1,1}$. Now we consider $d_6 \geq 5$. It is easy to see that $S = (5^6)$. Obviously, (5^6) is potentially $K_{3,1,1}$ -graphic.

Case: $n = 7$. First we assume that $\sigma(S) = 26$. Suppose $d_7 \leq 2$ and let S' be the degree sequence of $G - v_7$, so $\sigma(S') \geq 26 - 2 \times 2 = 22$. Then S' has a realization

containing a $K_{3,1,1}$ or $S' = (4^6)$. Therefore S has a realization containing a $K_{3,1,1}$ or $S = (5^1, 4^5, 1^1)$. Obviously, $(5^1, 4^5, 1^1)$ is potentially $K_{3,1,1}$ -graphic. In either event, S has a realization containing a $K_{3,1,1}$. Now we assume that $d_7 \geq 3$. It is easy to see that S is one of $(6^1, 5^1, 3^5)$, $(6^1, 4^2, 3^4)$, $(5^2, 4^1, 3^4)$, $(5^1, 4^3, 3^3)$ or $(4^5, 3^2)$. Obviously, all of them are potentially $K_{3,1,1}$ -graphic. Next, if $\sigma(S) = 28$, Suppose $d_7 \leq 3$. Let S' be the degree sequence of $G - v_7$, so $\sigma(S') \geq 28 - 3 \times 2 = 22$. Then S' has a realization containing a $K_{3,1,1}$ or $S' = (4^6)$. Therefore S has a realization containing a $K_{3,1,1}$ or $S = (5^2, 4^4, 2^1)$. Obviously, $(5^2, 4^4, 2^1)$ is potentially $K_{3,1,1}$ -graphic. In either event, S has a realization containing a $K_{3,1,1}$. Now we assume that $d_7 \geq 4$, then $S = (4^7)$. Clearly, (4^7) has a realization containing a $K_{3,1,1}$. Finally, suppose that $\sigma(S) \geq 30$. If $d_7 \leq 4$. Let S' be the degree sequence of $G - v_7$, so $\sigma(S') \geq 30 - 2 \times 4 = 22$. Then S' has a realization containing a $K_{3,1,1}$ or $S' = (4^6)$. Therefore S has a realization containing a $K_{3,1,1}$ or $S = (5^3, 4^3, 3^1)$. Clearly, $(5^3, 4^3, 3^1)$ has a realization containing a $K_{3,1,1}$. In either event, S has a realization containing a $K_{3,1,1}$. Now we consider $d_7 \geq 5$. It is easy to see that $\sigma(S) \geq 5 \times 7 = 35$. Obviously $\sigma(S) \geq 36$. Clearly, S has a realization containing a $K_{3,1,1}$.

We proceed by induction on n . Take $n \geq 8$ and make the inductive assumption that for $7 \leq t < n$, whenever S_1 is a t -term graphical sequence such that

$$\sigma(S_1) \geq 4t - 2$$

then S_1 has a realization containing a $K_{3,1,1}$. Let S be an n -term graphical sequence with $\sigma(S) \geq 4n - 2$. If $d_n \leq 2$, let S' be the degree sequence of $G - v_n$. Then $\sigma(S') \geq 4n - 2 - 2 \times 2 = 4(n - 1) - 2$. By induction, S' has a realization containing a $K_{3,1,1}$. Therefore S has a realization containing a $K_{3,1,1}$. Hence, we may assume that $d_n \geq 3$. By Proposition 2 and Theorem 4 of [5] (or Theorem 3.3 of [7]) S has a realization containing a K_4 . By Lemma 1 of [5], there is a realization G of S with v_1, v_2, v_3, v_4 , the four vertices of highest degree containing a K_4 . If $d(v_2) = 3$, then $4n - 2 \leq \sigma(S) \leq n - 1 + 3(n - 1) = 4n - 4$. This is a contradiction. Hence, we may assume that $d(v_2) \geq 4$. Let v_1 be adjacent to v_2, v_3, v_4, y_1 . If y_1 is adjacent to one of v_2, v_3, v_4 , then G contains a $K_{3,1,1}$. Hence, we may assume that y_1 is not adjacent to v_2, v_3, v_4 . Let v_2 be adjacent to v_1, v_3, v_4, y_2 . If y_2 is adjacent to one of v_1, v_3, v_4 , then G contains a $K_{3,1,1}$. Hence, we may assume that y_2 is not adjacent to v_1, v_3, v_4 . Since $d(y_1) \geq d_n \geq 3$, there is a new vertex y_3 , such that $y_1 y_3 \in E(G)$.

Case 1: Suppose $y_3 v_3 \in E(G)$. If $y_3 v_4 \in E(G)$, then G contains a $K_{3,1,1}$. Hence, we may assume that $y_3 v_4 \notin E(G)$. Then the edge interchange that removes the edges $y_1 y_3, v_3 v_4$ and $v_2 y_2$ and inserts the edges $y_1 v_2, y_3 v_4$ and $y_2 v_3$ produces a realization G' of S containing a $K_{3,1,1}$.

Case 2: Suppose $y_3 v_3 \notin E(G)$. Then the edge interchange that removes the edges $y_1 y_3, v_3 v_4$ and $v_2 y_2$ and inserts the edges $y_1 v_2, y_3 v_3$ and $y_2 v_4$ produces a realization G' of S containing a $K_{3,1,1}$. This finishes the inductive step, and thus Theorem 2 is established.

We make the following conjecture:

Conjecture.

$$\sigma(K_{p,1,1}, n) = 2[(p + 1)(n - 1) + 2]/2$$

for $n \geq 2p + 4$.

This conjecture is true for $p = 1$, by Theorem 3.5 of [4], for $p = 2$, by Theorem 1 of [6], and for $p = 3$, by the above Theorem 2.

Acknowledgment

The author thanks Prof. Therese Biedl for her valuable suggestions. The author thanks the referees for many helpful comments.

References

- [1] B. Bollobás, Extremal Graph Theory, Academic Press, London, 1978.
- [2] F. Harary, Graph Theory, Addison-Wesley Publishing Co., 1969
- [3] P. Erdős, On sequences of integers no one of which divides the product of two others and some related problems, *Izv. Naustno-Issl. Mat. i Meh. Tomsk* 2(1938), 74-82.
- [4] P. Erdős, M.S. Jacobson and J. Lehel, Graphs realizing the same degree sequences and their respective clique numbers, in *Graph Theory, Combinatorics and Application*, Vol. 1 (Y. Alavi et al., eds.), John Wiley and Sons, Inc., New York, 1991, 439-449.
- [5] R.J. Gould, M.S. Jacobson and J. Lehel, Potentially G -graphic degree sequences, in *Combinatorics, Graph Theory and Algorithms*, Vol. 2 (Y. Alavi et al., eds.), New Issues Press, Kalamazoo, MI, 1999, 451-460.
- [6] Lai Chunhui, A note on potentially $K_4 - e$ graphical sequences, *Australasian J. of Combinatorics* 24(2001), 123-127.
- [7] Li Jiong-Sheng and Song Zi-Xia, An extremal problem on the potentially P_k -graphic sequences, *Discrete Math.* (212)2000, 223-231.
- [8] Li Jiong-Sheng and Song Zi-Xia, The smallest degree sum that yields potentially P_k -graphical sequences, *J. Graph Theory*, 29(1998), 63-72.
- [9] Li Jiong-sheng and Song Zi-Xia, On the potentially P_k -graphic sequences, *Discrete Math.* 195(1999), 255-262.
- [10] Li Jiong-sheng, Song Zi-Xia and Luo Rong, The Erdős-Jacobson-Lehel conjecture on potentially P_k -graphic sequence is true, *Science in China(Series A)*, 41(5)(1998), 510-520.
- [11] Rong Luo, On potentially C_k -graphic sequences, *Ars Combinatoria* 64(2002), 301-318.

- [12] P. Turán, On an extremal problem in graph theory, Mat. Fiz. Lapok 48(1941), 436-452.
- [13] Yin Jian-Hua and Li Jiong-Sheng , An extremal problem on potentially $K_{r,s}$ -graphic sequences, Discrete Math, (260)2003, 295-305.